Journal of Statistical Physics, Vol. 94, Nos. 5/6, 1999

Hydrodynamics for Quasi-Free Quantum Systems

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Received January 22, 1998; final October 21, 1998

We consider quasi-free quantum systems and we derive the Euler equation using the so-called hydrodynamic limit. We use Wigner's well-known distribution function and discuss an extension to band distribution functions for particles in a periodic potential. We investigate the bosonic system of hard rods and calculate fluctuations of the density.

KEY WORDS: Euler equation; quantum distribution function; hydrodynamic limit.

1. INTRODUCTION

The Euler equations appear in hydrodynamics as scaling limits for the dynamics of the conserved quantities in a fluid. So far the only rigorous example of a microscopic derivation starting from Newton's laws is that of the one-dimensional hard rod fluid.[1–4] In this degenerate situation the number of particles with a given momentum is locally conserved. Therfore the hydrodynamic field $n_v(x, t)$ is just counting the number of particles at time t with space coordinate x and with momenta equal to v (we have put the masses equal to one). When the rod length tends to zero we deal with a one-dimensional ideal gas and, after the appropriate (Eulerian) scaling, the associated density field converges to the solution of

$$\partial_t f(x, v; t) + v\, \partial_x f(x, v; t) = 0 \tag{1.1}$$

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0022-4715/99/0300-0893$16.00/0 © 1999 Plenum Publishing Corporation

(for certain initial conditions). One can then go further and study the fluctuations around this macroscopic equation. They are Gaussian with a covariance matrix containing information about the transport coefficients which are finite and non-zero for the hard rod fluid.

In the present paper we investigate what remains of this in the quantum case. Then also here we pick out the simplest possible quantum dynamics and we simply ask how the corresponding Euler equations and the fluctuations around it can be derived. We see this as a modest step in the rigorous study of what could be called quantum hydrodynamics. Clealy, a mathematical derivation *ab initio* of the macroscopic equations of so-called quantum liquids is beyond reach. We may however learn something about the conceptual set-up to start such a project from a rigorous study in the most simple cases. We refer to refs. 3 and 5 for an overview and some solutions to some of the questions in the quantum domain of hydrodynamics. In all cases the underlying philosophy is however not fundamentally different from the classical situation. Conserved quantities vary on a much larger time scale than the others and by a law of large numbers the macroscopic equations appear as closed equations governing the motion of the rescaled conserved quantities.

Here we study quasi-free systems. That means that the microscopic dynamics is in some sense linear (as for classical Gaussian systems) and the equations are determined by the one-particle motion. Therefore we do not have to deal with the most important problem of time-scale separation (the Boltzmann–Gibbs principle) to see how the rescaling effectively truncates the hierarchy of kinetic equations. In this situation two papers by Herbert Spohn[5, 6] show how to get the program started. In the next section we summarize some of the ideas. In Sections 3 and 4 we discuss the derivation of the Euler equation (similar to (1.1)) for free systems subject to a slowly varying external potential. It is interesting to see that this can be applied (in Section 5) in the case of hard core bosons on the one-dimensional lattice. Section 6 deals with the non-equilibrium fluctuations. We compute the covariance of the rescaled density field. As expected, only at that moment do we start seeing a difference between the quantum statistics. Finally, in Section 7, we turn to the case of non-interacting quantum particles in a periodic potential.

2. SOME PRELIMINARIES

The first step in doing anything related to quantum hydrodynamics is probably asking for what quantum systems we can get a good idea of what are the conserved quantities and what is the structure of the equilibrium states. The answer is disappointing: Compared to the situation in so-called classical statistical mechanics, there is very little we know about truly interacting quantum systems. It is therefore not unreasonable to turn first to quasi-free systems where already on the microscopic scale the dynamics is governed by a one-particle Hamiltonian. A quasi-free dynamics is a quantum dynamics generated by a Hamiltonian H which is quadratic in the creation and annihilation operators $a^*(x)$, $a(x)$ of a Bose or Fermi field. Formally,

$$H = \int dx\, dy\, h_\varepsilon(x, y)\, a^*(x)\, a(y) \tag{2.1}$$

As usual with a varying number of particles, this should be thought of as acting on Fock space with single particle space $L^2(\mathbf{R}^d, dx)$. The quadratic form is specified by a one-particle Hamiltonian h_ε which in the physically more interesting situations has the form

$$h_\varepsilon = \tfrac{1}{2}[\vec{p} + e\vec{A}(\varepsilon x)]^2 + U(x) + V(\varepsilon x) \tag{2.2}$$

where U is a periodic potential, V is an external potential and $\vec{A}$ is the vector potential. The parameter $\varepsilon > 0$ specifies the scale over which the external potentials are varying and will be our scaling parameter in what follows. In the next section we start with the case where $U = 0$ but allowing a bit more general form of $h_\varepsilon(x, y)$ than obtained from (2.2). It is not too difficult to see that, when dealing with a quantum dynamics generated by (2.1), the evolution of the correlation functions can be specified in terms of the single particle evolution generated by h_ε. Therefore, the kinetic equations are closed even on the microscopic level which is an enormous advantage for doing hydrodynamics.

Notice however that the form (2.1) need not be restricted to the microscopic domain (where (2.1) is rather simplistic); there are plenty of quasi-free models in mesoscopic physics where (2.1) appears as effective action or as describing the evolution of quasi-particles. E.g., the central question in Bogoliubov's work on superfluidity was to see how to arrive at something like (2.1) (with some very particular properties) starting from a realistic interaction between He^4-particles.[7, 8] This is just a question of equilibrium statistical mechanics. Once arrived at (2.1) with an energy spectrum $E(k) \sim |k|$ for small $|k|$, Landau's theory (as, e.g., in ref. 9) takes over and completes the dynamical picture of superfluidity.

A quasi-free state is a state on the CAR or CCR algebra (the algebra's generated by the identity and the $a^*(x)$, $a(x)$ satisfying the Canonical (Anti)Commutation Relations) for which all correlations functions (and hence the state itself) are determined by the two-point functions, see e.g., ref. 10. For gauge invariant quasi-free states ω we must only specify the $\omega(a^*(x)\, a(y))$, which, in turn, are given in terms of a self-adjoint operator on the one-particle space.

Because of the dependence on the one-particle evolution in our systems (as explained above) it is crucial to understand the scaling of the long time behavior of a quantum particle subject to (2.2). To keep it simple, let us for a moment put $A=0$ in (2.2). Consider the position operator $r(t)$ in the Heisenberg picture and define its rescaling as

$$r^{\varepsilon}(t)=\varepsilon r(\varepsilon^{-1}t) \tag{2.3}$$

Rescaling the momentum operator as $p^{\varepsilon}(t)=p(\varepsilon^{-1}t)$ we see that the commutator $[r^{\varepsilon}, p^{\varepsilon}]$ vanishes as ε goes to zero. Let us therefore consider the semiclassical equations of motion, cf. refs. 11–13 and 6. Writing $E_n(k)$ ($n=1, 2,...$ and $k\in$ the Brillouin zone), for the (Bloch-)eigenvalues of the periodic part $h^o=\frac{1}{2}p^2+U(x)$ (with periodic boundary conditions), for each band the semiclassical equations ($\hbar=1$) look as follows:

$$\partial_t r=\partial_k E_n(k) \tag{2.4}$$

$$\partial_t k=-\partial_r V(r) \tag{2.5}$$

If at time $t=0$ the particle has a probability distribution $f_n(r, k)$, then, via (2.4)–(2.5), the distribution at time $t\geqslant 0$ is given by $f_n(r, k; t)=f_n(r(-t), k(-t))$ and solves

$$\partial_t f_n(r, k; t)+\partial_k E_n(k)\, \partial_r f_n(r, k; t)=\partial_r V(r)\, \partial_k f_n(r, k; t) \tag{2.6}$$

We can therefore expect that (2.6) gives the correct Euler equation for the considered quasi-free system but we still need to understand what to use for distribution function $f_n(r, k; t)$.

The next question is thus to see what is the analogue of classical quantities like $n_v(x, t)$ or $f(x, v; t)$ appearing in (1.1). Let us therefore briefly recall the notion of Wigner distribution function (cf. ref. 14). A recent and to this work very relevant mathematical survey is contained in Section 1 of ref. 11.[4] If $\psi(r)$ is the wave function of a quantum system, then we call ($h=1$):

$$F(r, k)=\left(\frac{1}{\pi}\right)^d \int_{\mathbf{R}^d} d\eta\, e^{2ik\cdot\eta}\, \bar{\psi}(r+\eta)\, \psi(r-\eta) \tag{2.7}$$

[4] We thank the referee for drawing our attention to this work.

the distribution function of the simultaneous values of the coordinates r and momenta k. Even though F is not positive in general (but it is real valued), this is traditionally justified by the following properties:

$$|\psi(r)|^2 = \int dk\, F(r,k) \tag{2.8}$$

$$|\tilde{\psi}(k)|^2 = \int dr\, F(r,k) \tag{2.9}$$

By ~ we denote Fourier transformation.

We wish to do exactly the same thing for our quasi-free systems. We will come back to the case of a periodic potential in Section 7 but for the moment we now also put $U=0$. Now we have, for every $\varepsilon > 0$ small, a state ω^ε and we must rescale space and time by ε^{-1}. Let α_t denote the microscopic time evolution to be generated by a quasi-free Hamiltonian and write the time-evolved state as $\omega^\varepsilon \circ \alpha_{\varepsilon^{-1}t} = \omega^\varepsilon_{\varepsilon^{-1}t}$. Then we define (cf. ref. 5),

$$f^\varepsilon(r,k;t) = \left(\frac{1}{\pi}\right)^d \left(\sum_\eta\right) \int d\eta\, e^{2i\eta\cdot k} \omega^\varepsilon_{\varepsilon^{-1}t}[a^*(\varepsilon^{-1}r+\eta)\, a(\varepsilon^{-1}r-\eta)] \tag{2.10}$$

as the macroscopic (scale ε^{-1}) one-particle distribution function of the system. Clearly, clustering conditions on the state ω^ε will be required to allow the convergence of the integral (sum) over η. From now on the assumption stands that this clustering holds allowing a well defined (2.10). Even though we write an integral in (2.10) we prefer to avoid here technicalities related to working in the continuum. We therefore mostly think of lattice systems (in which case we really have a discrete sum, $\varepsilon^{-1}r$ should be replaced by its integral part and the wave vector k is in the first Brillouin zone). Expressions like $a^*(x)\, a(y)$ should be understood in the distributional sense. Also later we will use the integral sign as a common symbol even when considering discrete systems. The analogous properties to (2.8)–(2.9) apply. As an example, note that if we are interested say in the density and if initially (at $t=0$) we have a product state with $\omega^\varepsilon[a^*(\varepsilon^{-1}r+\eta)\, a(\varepsilon^{-1}r-\eta)] = \delta(\eta)\, \rho(r)$, then $f^\varepsilon(r,k;0) = (1/2\pi)\, \rho(r)$ is the initial particle density.

The main question is now to investigate the limiting behavior (as ε goes to zero) of (2.10). That is, to derive what corresponds to the Euler equation (1.1). Aferwards we look for the fluctuations around this limiting behavior.

It should be clear by now that we do not identify the problem of deriving "quantum" Euler equations or of studying quantum hydrodynamics with that of "quantizing" the classical hydrodynamic equations. The macroscopic equations remain just "ordinary" partial differential equations for conserved quantities (and not for operators). Still, the quantum nature of the underlying system may in principle have a non-trivial effect on these equations (as for quantum liquids, superfluidity, etc.); if not on the macroscopic equations themselves then on the fluctuations (with corresponding transport properties) around it.

Scaling limits for Wigner functions (measures) are studied in the mathematics literature (see for instance the recent work by Gerard *et al.*[11] The starting point there is the Schrödinger equation

$$\partial_t \psi^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t) + ih_\varepsilon \psi^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t) = 0 \tag{2.11}$$

with $\psi^\varepsilon(x, t=0) = \psi^\varepsilon(x)$.[5] h_ε is the same type of Hamiltonian operator as in our quasi-free systems. In fact, we shall use their result to show the existence of $f(r, k; t)$, the limit of (2.10) as ε goes to zero. Moreover, to a large extent hydrodynamics of quasi-free quantum systems reduces to the homogenization limits considered in ref. 11.

For an introduction to semi-classical analysis we mention ref. (15).

3. EULER EQUATION FOR QUASI-FREE SYSTEMS

We consider a system of bosons or of fermions with formal Hamiltonian

$$H = \int dx\, dy\, h_\varepsilon(x, y)\, a^*(x)\, a(y)$$

As usual, the field of creation and annihilation operators is denoted by $a^*(x)$, $a(x)$. The kernel $h_\varepsilon(x, y)$ corresponds to a one-particle Hamiltonian. We always assume here that $h_\varepsilon(x, y) = h(\varepsilon(x+y)/2, y-x)$ with an appropriate decay condition, e.g., $\sup_x \int dy\, |h_\varepsilon(x, y)| < \infty$ for fermions, to generate an infinite volume quasi-free time evolution further denoted by α_t. Or put differently, the matrix elements $h_\varepsilon(x-y/2, x+y/2) = h(\varepsilon x, y)$ may be thought of as hopping rates (from place $x-y/2$ to $x+y/2$ by a distance y) varying slowly with x. The time t is real. Equivalently, we may write the Hamiltonian H via the Fourier transform $\tilde{a}(p) = (1/2\pi)^{d/2} \int du\, e^{ipu} a(u)$,

$$H = \left(\frac{1}{2\pi}\right)^d \int dk \int dp \int du\, e^{i(k-p)u} E(\varepsilon u, (k+p)/2)\, \tilde{a}^*(k)\, \tilde{a}(p)$$

[5] Regarding notation, we identify the $u^\varepsilon(\varepsilon x, \varepsilon t)$ of ref. 11 with our $\psi^\varepsilon(x, t)$.

where the energy spectrum, E, is defined by: $E(r, k) = \int dv\, e^{-ikv} h(r, v)$. Clearly, in the limit of infinite scale separation, ε to zero, the microscopic time evolution is translation invariant with (true) energy spectrum $E(k) = E(0, k)$. Yet, as we will see shortly, the (macroscopic) Euler equation remembers the dependence of $E(r, k)$ on r. Finally, the simplest example corresponds to a free system in a slowly varying chemical potential, $E(r, k) = E(k) + V(r)$, which corresponds to

$$H = \int dk\, E(k)\, \tilde{a}^*(k)\, \tilde{a}(k) + \int du\, V(\varepsilon u)\, a^*(u)\, a(u)$$

In order to find conditions under which the existence of the limiting f (see (2.10)) is guaranteed we recall the work by Gerard *et al.*[11] Suppose the 2-point (which is the only relevant information on the state) function of ω^ε is of the (general) form

$$\omega^\varepsilon[a^*(x)\, a(y)] = \int d\mu(\lambda)\, \bar{\psi}^\varepsilon_\lambda(x)\, \psi^\varepsilon_\lambda(y) \tag{3.1}$$

where ψ^ε_λ satisfies the Schrödinger equation (2.11). $d\mu(\lambda)$ is an absolutely integrable signed measure. On h_ε and ψ^ε_λ we assume the same conditions as in ref. 11. This decomposition includes the closed convex hull of pure states (indexed by λ). Note, that

$$f^\varepsilon(r, k; t) = \int d\mu(\lambda)\, w^\varepsilon[\psi^\varepsilon_\lambda](x, k; t) \tag{3.2}$$

where $w^\varepsilon[\psi^\varepsilon_\lambda](x, k; t)$ is the Wigner function used in ref. 11. This relation provides an alternative way to derive the Euler equation by quoting their result on $w^\varepsilon[\psi^\varepsilon_\lambda](x, k; t)$.

We write $f(r, k; t)$ for any limit $f(r, k; t) = \lim_{\varepsilon \downarrow 0} f^\varepsilon(r, k; t)$, $\partial_s f(r, k; t) = \lim_{\varepsilon \downarrow 0} \partial_s f^\varepsilon(r, k; t)$, $s = t, r, k$ and we wish to see what equation is satisfied by such a limiting function.

Proposition 1. Under the above hypotheses.

$$\partial_t f(r, k; t) + \partial_k E(r, k)\, \partial_r f(r, k; t) = \partial_r E(r, k)\, \partial_k f(r, k; t) \tag{3.3}$$

Proof. There are different ways to verify (3.3). Below we present an explicit and detailed computation.

$$\partial_t f^\varepsilon(r,k;t)$$

$$=\left(\frac{1}{2\pi}\right)^d \int d\eta\, e^{i\eta k}\,\partial_t\left[\omega^\varepsilon_{\varepsilon^{-1}t}\left[a^*\left(\varepsilon^{-1}r+\frac{\eta}{2}\right)a\left(\varepsilon^{-1}r-\frac{\eta}{2}\right)\right]\right]$$

$$=\frac{-i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{3d}\int d\eta\, e^{i\eta k}\int dp\, dp'\, e^{ip\varepsilon^{-1}r-ip'\varepsilon^{-1}r+i(n/2)(p+p')}\int dx\, dy\, dz$$

$$\times e^{-ipx+ip'y}\left[h_\varepsilon(x,z)\,\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(z)\,a(y)]-\overline{h_\varepsilon(y,z)}\,\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(x)\,a(z)]\right]$$

$$=\frac{-i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{3d}\int d\eta\, dp\, dp'\, e^{i\eta(k+(p+p')/2)+i(p-p')\,\varepsilon^{-1}r}$$

$$\times\left[\int dy\, dn\, dm\, e^{ip'y-ipm-ipn}\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(m)\,a(y)]\,h_\varepsilon(n+m,m)\right.$$

$$\left.-\int dx\, dn\, dm\, e^{-ipx+ip'm+ip'n}\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(x)\,a(m)]\,\overline{h_\varepsilon(n+m,m)}\right]$$

$$=2^d\,\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{3d}\int d\eta\, du\, dv\, e^{i\eta(k+u)+2iv\varepsilon^{-1}r}$$

$$\times\left[\int dy\, dn\, dm\, e^{i(u-v)\,y-i(u+v)(n+m)}\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(m)\,a(y)]\,h_\varepsilon(n+m,m)\right.$$

$$\left.-\int dx\, dn\, dm\, e^{-i(u+v)\,x+i(u-v)(n+m)}\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(x)\,a(m)]\,\overline{h_\varepsilon(n+m,m)}\right]$$

$$=2^d\,\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{2d}\int dy\, dn\, dm\,\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(m)\,a(y)]\,h_\varepsilon(n+m,m)$$

$$\times\int du\, dv\,\delta(k+u)\,e^{-iu(n+m-y)}e^{2iv(\varepsilon^{-1}r-(n+m+y)/2)}$$

$$-2^d\,\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{2d}\int dx\, dn\, dm\,\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(x)\,a(m)]\,\overline{h_\varepsilon(n+m,m)}$$

$$\times\int du\, dv\,\delta(k+u)\,e^{iu(n+m-x)}e^{2iv(\varepsilon^{-1}r-(n+m+x)/2)}$$

$$=\frac{-i}{\varepsilon}\left(\frac{1}{2\pi}\right)^{d}\int dy\, dn\, dm\,\omega^\varepsilon_{\varepsilon^{-1}t}[a^*(m)\,a(y)]\,h_\varepsilon(n+m,m)$$

$$\times e^{ik(n+m-y)}\delta\left(\varepsilon^{-1}r-\frac{n+m+y}{2}\right)$$

$$+\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^d \int dx\, dn\, dm\, \omega^{\varepsilon}_{\varepsilon^{-1}t}[a^*(x)\, a(m)]\, \overline{h_\varepsilon(n+m, m)}$$

$$\times e^{-ik(n+m-x)}\delta\left(\varepsilon^{-1}r - \frac{n+m+x}{2}\right)$$

$$=\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^d \int dn\, d\bar{y}\, d\eta\, \omega^{\varepsilon}_{\varepsilon^{-1}t}\left[\left(a^*\left(\bar{y}+\frac{\eta}{2}\right) a\left(\bar{y}-\frac{\eta}{2}\right)\right]$$

$$\times h_\varepsilon\left(n+\bar{y}+\frac{\eta}{2}, \bar{y}+\frac{\eta}{2}\right) e^{ik(n+\eta)}\delta\left(\varepsilon^{-1}r - \frac{n}{2} - \bar{y}\right)$$

$$-\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^d \int dn\, d\bar{x}\, d\eta\, \omega^{\varepsilon}_{\varepsilon^{-1}t}\left[\left(a^*\left(\bar{x}+\frac{\eta}{2}\right) a\left(\bar{x}-\frac{\eta}{2}\right)\right]$$

$$\times \overline{h_\varepsilon\left(n+\bar{x}-\frac{\eta}{2}, \bar{x}-\frac{\eta}{2}\right)}\, e^{-ik(n-\eta)}\delta\left(\varepsilon^{-1}r - \frac{\eta}{2} - \bar{x}\right)$$

$$=\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^d \int dn\, d\eta \left[e^{ik(n+\eta)}\omega^{\varepsilon}_{\varepsilon^{-1}t}\left[a^*\left(\varepsilon^{-1}r - \frac{n}{2} + \frac{\eta}{2}\right) a\left(\varepsilon^{-1}r - \frac{n}{2} - \frac{\eta}{2}\right)\right]\right.$$

$$\times h_\varepsilon\left(\varepsilon^{-1}r + \frac{n+\eta}{2}, \varepsilon^{-1}r - \frac{n-\eta}{2}\right)$$

$$-\frac{i}{\varepsilon}\left(\frac{1}{2\pi}\right)^d \int dn\, d\eta\, e^{-ik(n-\eta)}\omega^{\varepsilon}_{\varepsilon^{-1}t}\left[a^*\left(\varepsilon^{-1}r - \frac{n}{2} + \frac{\eta}{2}\right)\right.$$

$$\left.\times a\left(\varepsilon^{-1}r - \frac{n}{2} - \frac{\eta}{2}\right)\right] \overline{h_\varepsilon\left(\varepsilon^{-1}r + \frac{n-\eta}{2}, \varepsilon^{-1}r - \frac{n+\eta}{2}\right)}\Bigg]$$

Now, inserting the definitions of $f^\varepsilon(r, k; t)$ and $h(r, n)$ into the last expression and expanding both at r, we find that

$$\lim_{\varepsilon \downarrow 0} \partial_t f^\varepsilon(r, k; t)$$

$$=\left(\frac{1}{2\pi}\right)^d \lim_{\varepsilon\downarrow 0} \frac{i}{\varepsilon} \int dn\, d\eta\, dv\, e^{ik\eta - i\eta v} f^\varepsilon\left(r - \varepsilon\, \frac{n}{2}, v; t\right)$$

$$\times\left[e^{ikn} h\left(r + \varepsilon\, \frac{\eta}{2}, -n\right) - e^{-ikn} h\left(r - \varepsilon\, \frac{\eta}{2}, n\right)\right]$$

$$= \left(\frac{1}{2\pi}\right)^d \lim_{\varepsilon \downarrow 0} \frac{i}{\varepsilon} \int d\eta\, dv\, e^{ik\eta - i\eta v} f^\varepsilon(r, v; t) \int dn (e^{ikn} h(r, -n) - e^{-ikn} h(r, n))$$

$$+ \left(\frac{1}{2\pi}\right)^d \lim_{\varepsilon \downarrow 0} \int d\eta\, dv\, e^{ik\eta - i\eta v}\, \partial_r f^\varepsilon(r, v; t) \int dn\, ine^{-ikn} h(r, n)$$

$$+ \left(\frac{1}{2\pi}\right)^d \lim_{\varepsilon \downarrow 0} \int d\eta\, dv\, i\eta e^{ik\eta - i\eta v} f^\varepsilon(r, v; t)\, \partial_r \int dn\, e^{-ikn} h(r, n)$$

$$= -\partial_k E(r, k)\, \partial_r f(r, k; t) + \partial_r E(r, k)\, \partial_k f(r, k; t) \quad \blacksquare$$

4. REMARKS

1. Proposition 1 can be extended to time dependent quasi-free time evolutions, $U(t, 0) = T[e^{-i \int_0^t dt'\, h_\varepsilon(t')}]$, where $h_\varepsilon(t)$ is a family of self-adjoint operators such that the integral gives a well-defined self-adjoint operator. T is the time ordering operation. If $h_\varepsilon(x, y; t) = h(\varepsilon(x+y)/2, y-x; \varepsilon t)$ then the function $E(r, k)$ in Eq. (3.3) is simply replaced by an analogous function $E(r, k; t)$.

2. The general solution of (3.3) is given by

$$f(r, k; t) = f(r(-t), k(-t); 0) \tag{4.1}$$

$r(t)$, $k(t)$ are determined by the classical Hamilton's equations of motion:

$$\partial_t r = \partial_k E(r, k) \tag{4.2}$$

$$\partial_t k = -\partial_r E(r, k) \tag{4.3}$$

with Hamilton function $E(r, k)$ and $r(0) = r$, $k(0) = k$.

3. Analyzing a sum of (properly scaled) quasi-free Hamilton operators is reduced to the sum of their respective energy spectra; applicable for example to the case $h_\varepsilon = (1/2m)(\vec{p} + (e/c)\, \vec{A}(\varepsilon x))^2 + eV(\varepsilon x)$.

4. The analysis can easily be extended to generators, H, which are not neccessarily self-adjoint, but where the difference between H and its adjoint H^* is of order ε. As an example consider $h_\varepsilon = -\frac{1}{2}\Delta_\gamma$, generating a random walk with bias $\gamma = (\gamma_1, \ldots, \gamma_d)$ on the lattice $\mathbf{Z}^d$ (with unit vectors e_i in the positive lattice directions):

$$\Delta_\gamma f(x) = \sum_{i=1}^{d} [(1 + \gamma_i/2)\, f(x + e_i) + (1 - \gamma_i/2)\, f(x - e_i) - 2f(x)]$$

If $\gamma_i = \gamma(\varepsilon)$ be such that $\gamma_i(\varepsilon)/\varepsilon \to a_i$ as $\varepsilon \downarrow 0$. Then, $f(r, k; t)$ satisfies

$$\partial_t f(r, k; t) + \sin k \cdot [(\nabla + a) f(r, k; t)] = 0 \tag{4.4}$$

The velocity term comes from the real part of the dispersion relation $E(r, k) = 1 - \cos k$ whereas the drift a stems from the imaginary part of E.

5. Notice that the density of particles, $\rho(r, t)$, is given by

$$\rho(r, t) = \int dk\, f(r, k; t)$$

An example where one can say more about $\rho(r, t)$ is obtained from taking as the initial distribution the function $f(r, k) = (1/2\pi)\, \rho(r)$ with $k \in [-\pi, \pi]$. Take $\mathbf{Z}^1$ for the lattice case, $h = -\frac{1}{2}\Delta$ the lattice Laplacian as one particle Hamiltonian and the following initial "diagonal" states ω^ε:

$$\omega^\varepsilon[a^*(x)\, a(y)] = \delta(x, y) \cdot \rho(\varepsilon x)$$

Then, $\rho(r, t)$ satisfies the modified Bessel equation

$$[t^2 \partial_t^2 + t\partial_t - t^2 \Delta]\, \rho(r, t) = 0 \tag{4.5}$$

In other words, $\tilde{\rho}(p, t)$ is the (normal) Bessel function, which for $p \neq 0$ goes oscillating to 0 as $|t|$ tends to infinity.

5. QUANTUM HARD CORE BOSONS

In this section we consider a simple model of interacting bosons (cf. ref. 16). We discuss first an on-site hard core interaction on $\mathbf{Z}^1$ (i.e., hard core radius 0), which has the exclusion effect that at most one boson occupies a single lattice site. Later we explain extensions to positive core length.

Let $H_\Lambda(\lambda)$ be the following bosonic Hamilton operator on an interval $\Lambda \subset \mathbf{Z}$ with point interaction $V(x, y) = \lambda \cdot \delta(x, y)$ of strength λ and free end boundary conditions,

$$H_\Lambda(\lambda) = -\frac{1}{2} \sum_{x \in \Lambda} b^*(x)(\Delta b)(x) + \lambda \sum_{x \in \Lambda} b^*(x)^2\, b(x)^2 \tag{5.1}$$

Here, $b^*(x)$, $b(y)$ denote the Bose field operators on $\mathbf{Z}$. $H_\Lambda(\lambda)$ acts on the usual bosonic Fock space, $\mathscr{F}(\Lambda)$. (Notice, that H_Λ is not self-adjoint but this has no effect for the time-evolution of local observables in the termodynamic limit.) The limit of $H_\Lambda(\lambda)$ as λ goes to infinity displays an interesting Hamilton operator. This limit is well-defined on states

$$|\psi_X\rangle = \prod_{x \in X} b^*(x) |0\rangle, \qquad X \subset \Lambda \tag{5.2}$$

which define the Fock space, $\mathscr{F}^{hc}(\Lambda)$, of hard core bosons. $|0\rangle$ is the vacuum state. Let P: $\mathscr{F}(\Lambda) \to \mathscr{F}^{hc}(\Lambda)$ be the orthogonal projection onto the hard core boson states. We introduce the following annihilation (creation) operators

$$a^{(*)}(x) = Pb^{(*)}(x) P \tag{5.3}$$

In particular, they have the following properties:

1. The state space $\mathscr{F}^{hc}(\Lambda)$ is invariant under $b(x)$ and therefore $a(x) = b(x) P$.

2. $a^*(x)$, $a(y)$ satisfy mixed (Anti)Commutation relations:

$$[a^*(x), a(y)]_- = 0, \qquad \text{for} \quad x \neq y \tag{5.4}$$

$$[a^*(x), a(x)]_+ = 1 \tag{5.5}$$

where $[\ ,\ ]_-$, $[\ ,\ ]_+$ denote the Anti-, commutator, respectively.

3. $\lim_{\lambda \to \infty} H_\Lambda(\lambda) = -\frac{1}{2} \sum_{x \in \Lambda} a^*(x)(\Delta a)(x)$ on $\mathscr{F}^{hc}(\Lambda)$.

The commutation relations are familiar from spin systems. Although the previous formulation works in any dimension the statistics of the $a^\#(x)$'s can only be efficiently "repaired" in one dimension. By applying the Klein–Jordan–Wigner transformation (cf. ref. 17) we map the hard core Bose field operators onto Fermi field operators. Let

$$c_\Lambda(x) = e^{-i\pi \sum_{j \leqslant x-1} a^*(j)\, a(j)} a(x) \tag{5.6}$$

$$c^*_\Lambda(x) = a^*(x)\, e^{i\pi \sum_{j \leqslant x-1} a^*(j)\, a(j)} \tag{5.7}$$

where the sum goes over all $j \in \Lambda$ left to x.

This is the usual Klein–Jordan–Wigner transformation. One can make sense of the formal limit $\Lambda \uparrow \mathbf{Z}$ (cf. ref. 10) and the resulting $c^*(x)$, $c(y)$ satisfy the usual Anticommutation relation. Observe that under the above transformation, formally,

$$H_{\mathbf{Z}} = -\frac{1}{2} \sum_{x \in \mathbf{Z}} a^*(x)(\Delta a)(x) = -\frac{1}{2} \sum_{x \in \mathbf{Z}} c^*(x)(\Delta c)(x) \tag{5.8}$$

is the free fermion Hamilton operator. We know that

$$n(r, k) = \sum_{\eta \in \mathbf{Z}} e^{2i\eta k} c^*(r-\eta)\, c(r+\eta) \tag{5.9}$$

are locally conserved quantities for the free force evolution. Rewriting them in terms of hard core bosons we get

$$n(r, k) = \sum_{\eta \in \mathbf{Z}} e^{2i\eta k} e^{-i\pi N(r-\eta,\, r+\eta-1)} a^*(r-\eta)\, a(r+\eta) \tag{5.10}$$

where $N(x, y) = \sum_{x \leqslant j \leqslant y} a^*(j)\, a(j)$. This formula is obtained from (5.9) via substituting Definitions (5.6)–(5.7) with $\Lambda \uparrow \mathbf{Z}$. In this way the hard core bosons are connected to a free fermion system but the Wigner distribution for the fermion density does of course not correspond to the Wigner transform of the boson density. Although the time evolution of the $a^{\#}(x)$'s is quite complicated (and unknown) we are only interested in a special combination appearing in $n(r, k)$. So once the transformation as in Definitions (5.6)–(5.7) is completed (with $\Lambda \uparrow \mathbf{Z}$), the problem is simply reduced to free fermions on the lattice (but only as far as this local quantity, $n(r, k)$, is concerned). It is then a simple matter to apply Proposition 1:

Proposition 2. Suppose that $f(r, k) = \lim_{\varepsilon \downarrow 0} \omega^{\varepsilon}[n(\varepsilon^{-1} r, k)]$ exists (for example if the functions $\psi_{\lambda}^{\varepsilon}$ in the decomposition of ω^{ε} (see 3.1) are ε-oscillatory (see ref. 11 and bounded). Then, under the dynamics generated by the Hamilton operator (5.8), $f(r, k; t)$ exists and satisfies the free force Euler equation

$$\partial_t f(r, k; t) + \sin k \cdot \partial_r f(r, k; t) = 0 \tag{5.11}$$

Remarks. 1. The same analysis works also for bosons with zero core radius in $\mathbf{R}^1$. It results in the free Euler equation $(\partial_t + k\,\partial_r)\, f(r, k; t) = 0$.

2. We can contract any (continuous or discrete) configuration of rods with positive (integral) length bijectively onto a configuration of points in a contracted configuration space, i.e., onto an above configuration. Therefore the proposition extends to hard core interactions (with *Dirichlet* boundary conditions) of positive length.

3. Imposing Neumann boundary conditions (for example) we expect a non-linear equation similar to the Euler equation for classical hard rods proven in ref. 1. This is an open problem.

6. FLUCTUATIONS OF THE DENSITY

The Euler equation corresponds mathematically to a law of large numbers. The simplest equation for the density profile is (4.5). Now we study (non-equilibrium) fluctuations of the particle density for the quasi-free systems of Section 3. The field of fluctuations, $\xi^\varepsilon(\psi, A, t)$, of a local observable, A, is defined by

$$\xi^\varepsilon(\psi, A, t) = \varepsilon^{d/2} \int dr\, \psi(\varepsilon r)[\tau_{r, \varepsilon^{-1} t}(A) - \omega^\varepsilon(\tau_{r, \varepsilon^{-1} t}(A))] \tag{6.1}$$

Here, ψ is any test function and ω^ε a family of states as in Section 2. Further, we denote by $\tau_{r,t}(A)$ the space and time shift of the observable, A, by r and t, respectively. We are interested in the density, $\tau_{r,t}(A) = \alpha_t(a^*(r)\, a(r))$, in particular in realizing the limit

$$\lim_{\varepsilon \downarrow 0} \xi^\varepsilon(\psi, A, t) = \int dr\, \psi(r)\, \xi(r, t) \tag{6.2}$$

We should therefore study the characteristic function, $\omega^\varepsilon(e^{i\lambda \xi^\varepsilon(\psi, A, t)})$, and try to reconstruct the limiting dynamics of the fluctuation field. Since we restrict our attention to quasi-free states (cf. ref. 10) and we always assume nice clustering properties it is fair to expect Gaussian behavior. The precise meaning of this in the quantum case (the so called quantum central limit theorem) can be found in refs. 18 and 19. We are satisfied therefore with studying the covariance $\lim_{\varepsilon \downarrow 0} \omega^\varepsilon[\xi^\varepsilon(\psi, t)\, \xi^\varepsilon(\phi, s)]$ with $\xi^\varepsilon(\phi, s)$ given by (6.1) for $A = a^*(0)\, a(0)$ and for a class of test functions $\psi, \phi \in \mathscr{D}$.

Clearly, the dynamics of the fluctuations can only be derived when the law of large number is already established. Therefore, while the next proposition does not follow directly from Proposition 1, it is essential to make sense of fluctuations that one first understands the averaged behavior. We put ourselves therefore in the same context as for Proposition 1 and we have that $f(r, k; t) = \lim_{\varepsilon \downarrow 0} f^\varepsilon(r, k; t)$. On the level of the one-particle dynamics we recall that we denote by $r^\varepsilon(t)$ the rescaled position operator, see (2.3), and we assume that it converges to an operator $r(t)$ as $\varepsilon \downarrow 0$, see refs. 13, 12, and 6. This convergence takes place on a suitable domain $\mathscr{D}$ dense in $L^2(\mathbf{R}^d, dx)$. Let us write

$$(e^{ir(t)}\psi)(x, k) = \left(\frac{1}{2\pi}\right)^{d/2} \int dq\, \tilde{\psi}(q)\, e^{iqr(x, k; t)} \tag{6.3}$$

where $r(x, k; t)$ is the solution of (4.2). (If the motion is force free then $(e^{ir(t)}\psi)(x, k) = \psi(x + \partial_k E(k)\, t)$.)

Proposition 3. Under the above conditions on the initial states and on the Hamilton operator,

$$\lim_{\varepsilon\downarrow 0} \omega^{\varepsilon}[\xi^{\varepsilon}(\psi,t)\,\xi^{\varepsilon}(\phi,s)]$$

$$= \int dx\, dk (e^{ir(t-s)}\psi)(x,k)\,\phi(x)\; f(x,k;s)(1 \pm f(x,k;s)) \qquad (6.4)$$

where $+/-$ concerns Fermi/Bose statistics, respectively.

Proof.

$$\omega^{\varepsilon}[\xi^{\varepsilon}(\psi,t)\,\xi^{\varepsilon}(\phi,s)]$$

$$= \varepsilon^d \int dr\, dr'\, \psi(\varepsilon r)\,\phi(\varepsilon r')$$

$$\times \{\omega^{\varepsilon}[a^*(r,\varepsilon^{-1}t)\,a(r,\varepsilon^{-1}t)\,a^*(r',\varepsilon^{-1}s)\,a(r',\varepsilon^{-1}s)]$$

$$-\omega^{\varepsilon}[a^*(r,\varepsilon^{-1}t)\,a(r,\varepsilon^{-1}t)]\,\omega^{\varepsilon}[a^*(r',\varepsilon^{-1}s)\,a(r',\varepsilon^{-1}s)]\}$$

$$= \varepsilon^d \int dr\, dr'\, \psi(\varepsilon r)\,\phi(\varepsilon r')\,\omega^{\varepsilon}[a^*(r,\varepsilon^{-1}t)\,a(r',\varepsilon^{-1}s)]$$

$$\times \omega^{\varepsilon}[a(r,\varepsilon^{-1}t)\,a^*(r',\varepsilon^{-1}s)]$$

By changing the order of a and a^* in the last expectation we get two parts. The one including the (anti)commutator is equal to

$$\varepsilon^d \int dr\, dr'\, \psi(\varepsilon r)\,\phi(\varepsilon r')\,\omega^{\varepsilon}[a^*(r,\varepsilon^{-1}t)\,a(r',\varepsilon^{-1}s)]$$

$$\times \omega^{\varepsilon}([a(r,\varepsilon^{-1}t), a^*(r',\varepsilon^{-1}s)]_{\pm})$$

$$= \left(\frac{1}{2\pi}\right)^{d/2} \varepsilon^d \int dr'\, dq\, \tilde{\psi}(q)\,\phi(\varepsilon r') \int dz\, dr\, e^{iq\varepsilon r} e^{-ih_{\varepsilon}(t-s)\varepsilon^{-1}}(r,z)$$

$$\times e^{ih_{\varepsilon}(t-s)\varepsilon^{-1}}(r',r)\,\omega^{\varepsilon}[a^*(z,\varepsilon^{-1}s)\,a(r',\varepsilon^{-1}s)]$$

$$= \left(\frac{1}{2\pi}\right)^{d/2} \varepsilon^d \int dr'\, dq\, \tilde{\psi}(q)\,(\varepsilon r') \int dz\, e^{iqr^{\varepsilon}(t-s)}(r',z)\,\omega^{\varepsilon}_{\varepsilon^{-1}s}[a^*(z)\,a(r')]$$

$$= \left(\frac{1}{2\pi}\right)^{d/2} \varepsilon^d \int dx\, d\eta\, dq\, \tilde{\psi}(q)\,\phi\left(\varepsilon\left(x-\frac{\eta}{2}\right)\right) e^{iqr^{\varepsilon}(t-s)}$$

$$\times \left(x-\frac{\eta}{2}, x+\frac{\eta}{2}\right)\omega^{\varepsilon}_{\varepsilon^{-1}s}\left[a^*\left(x+\frac{\eta}{2}\right)a\left(x-\frac{\eta}{2}\right)\right]$$

$$= \left(\frac{1}{2\pi}\right)^{d/2} \int dx\, d\eta\, dk\, dq\, \tilde{\psi}(q)\, \phi\left(x - \varepsilon\,\frac{\eta}{2}\right) e^{iqr^{\varepsilon}(t-s)}$$

$$\times \left(\varepsilon^{-1}x - \frac{\eta}{2}, \varepsilon^{-1}x + \frac{\eta}{2}\right) e^{-i\eta k} f^{\varepsilon}(x, k; s)$$

Now remember our scaling of the matrix elements of h_{ε} and take the limit $\varepsilon \downarrow 0$ of the last expression:

$$\left(\frac{1}{2\pi}\right)^{d/2} \int dx\, dk\, dq\, \tilde{\psi}(q)\, \phi(x) \left(\int d\eta\, e^{iqr(t-s)}(x, \eta)\, e^{-i\eta k}\right) f(x, k; s)$$

$$= \int dx\, dk (e^{ir(t-s)}\psi)(x, k)\, \phi(x)\; f(x, k; s)$$

For the second part (forgetting the $(2\pi)^{d/2}$) we need

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{d} \int dq\, dr'\, \tilde{\psi}(q)\, \phi(\varepsilon r') \int dr\, e^{iq\varepsilon r} \omega^{\varepsilon}_{\varepsilon^{-1}s}$$

$$\times [a^{*}(r, \varepsilon^{-1}(t-s))\, a(r')]\; \omega^{\varepsilon}_{\varepsilon^{-1}s}[a^{*}(r')\, a(r, \varepsilon^{-1}(t-s))]$$

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{d} \int dq\, dr'\, \tilde{\psi}(q)\, \phi(\varepsilon r') \int dr\, dz\, dz'\, e^{iq\varepsilon r} e^{-ih\varepsilon^{-1}(t-s)}(r, z)\, \omega^{\varepsilon}_{\varepsilon^{-1}s}$$

$$\times [a^{*}(z)\, a(r')]\, e^{ih_{\varepsilon}(t-s)\varepsilon^{-1}}(z', r)\, \omega^{\varepsilon}_{\varepsilon^{-1}s}[a^{*}(r')\, a(z')]$$

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{d} \int dq\, dr'\, \tilde{\psi}(q)\, \phi(\varepsilon r') \int dz\, dz'\, e^{iq\varepsilon r(\varepsilon^{-1}(t-s))}(z', z)\, \omega^{\varepsilon}_{\varepsilon^{-1}s}$$

$$\times [a^{*}(z)\, a(r')]\; \omega^{\varepsilon}_{\varepsilon^{-1}s}[a^{*}(r')\, a(z')]$$

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{d} \int dq\, dx\, d\eta\, \tilde{\psi}(q)\, \phi\left(\varepsilon\left(x - \frac{\eta}{2}\right)\right) \int dz'\, e^{iqr^{\varepsilon}(t-s)}$$

$$\times \left(z', x + \frac{\eta}{2}\right) \omega^{\varepsilon}_{\varepsilon^{-1}s}\left[a^{*}\left(x + \frac{\eta}{2}\right) a\left(x - \frac{\eta}{2}\right)\right] \omega^{\varepsilon}_{\varepsilon^{-1}s}\left[a^{*}\left(x - \frac{\eta}{2}\right) a(z')\right]$$

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{d} \int dq\, dx\, d\eta\, \tilde{\psi}(q)\, \phi(\varepsilon x) \int dz'\, dk\, e^{iqr^{\varepsilon}(t-s)}$$

$$\times \left(z', x + \frac{\eta}{2}\right) e^{-i\eta k} f^{\varepsilon}(\varepsilon x, k; s)\, \omega^{\varepsilon}_{\varepsilon^{-1}s}\left[a^{*}\left(x - \frac{\eta}{2}\right) a(z')\right]$$

$$
\begin{aligned}
&= \lim_{\varepsilon \downarrow 0} \varepsilon^d \int dq\, dy\, d\eta\, d\bar{\eta}\, dk\, d\bar{k}\, \tilde{\psi}(q)\, \phi\left(\varepsilon\left(y + \frac{\bar{\eta}}{2}\right)\right) e^{iqr^\varepsilon(t-s)} \\
&\quad \times \left(y - \frac{\bar{\eta}}{2}, y + \frac{\bar{\eta}}{2} + \eta\right) e^{-i\eta k - i\bar{\eta}\bar{k}} f^\varepsilon\left(\varepsilon\left(y + \frac{\bar{\eta}+\eta}{2}\right), k; s\right) f^\varepsilon(\varepsilon y, \bar{k}; s) \\
&= \lim_{\varepsilon \downarrow 0} \int dq\, dy\, d\eta\, d\bar{\eta}\, dk\, d\bar{k}\, \tilde{\psi}(q)\, \phi\left(y + \varepsilon\, \frac{\bar{\eta}}{2}\right) e^{iqr^\varepsilon(t-s)} \\
&\quad \times \left(\varepsilon^{-1}\left(y + \varepsilon\, \frac{\eta}{2}\right) - \frac{\eta+\bar{\eta}}{2}, \varepsilon^{-1}\left(y + \varepsilon\, \frac{\eta}{2}\right) + \frac{\eta+\bar{\eta}}{2}\right) \\
&\quad \times e^{-i\eta k - i\bar{\eta}\bar{k}} f^\varepsilon\left(y + \varepsilon\, \frac{\bar{\eta}+\eta}{2}, k; s\right) f^\varepsilon(y, \bar{k}; s) \\
&= \lim_{\varepsilon \downarrow 0} \int dq\, dy\, d\eta\, d\bar{\eta}\, dk\, d\bar{k}\, \tilde{\psi}(q)\, \phi\left(y + \varepsilon\, \frac{\bar{\eta}}{2}\right) e^{iqr(t-s)} \left(y + \varepsilon\, \frac{\eta}{2}, \frac{\eta+\bar{\eta}}{2}\right) \\
&\quad \times e^{-i\eta k - i\bar{\eta}\bar{k}} f^\varepsilon\left(y + \varepsilon\, \frac{\bar{\eta}+\eta}{2}, k; s\right) f^\varepsilon(y, \bar{k}; s) \\
&= \lim_{\varepsilon \downarrow 0} \int dq\, dy\, dv\, d\bar{v}\, dp\, d\bar{p}\, \tilde{\psi}(q)\, \phi(y)\, e^{iqr(t-s)}(y, v) \\
&\quad \times e^{-ivp - i\bar{v}\bar{p}} f^\varepsilon(y, p + \bar{p}; s)\; f^\varepsilon(y, p - \bar{p}; s) \\
&= (2\pi)^{d/2} \int dy\, d\bar{v}\, dp\, d\bar{p} (e^{ir(t-s)}\psi)(y, p)\, \phi(y) \\
&\quad \times e^{-i\bar{v}\bar{p}} f(y, p + \bar{p}; s)\; f(y, p - \bar{p}; s) \\
&= (2\pi)^{d/2} \int dy\, dp (e^{ir(t-s)}\psi)(y, p)\, \phi(y)\; f^2(y, p; s) \qquad \blacksquare
\end{aligned}
$$

7. PARTICLES IN A PERIODIC POTENTIAL

So far we have not dealt with the important case of particles moving in a periodic potential. More precisely, where the system is quasi-free with one particle Hamiltonian of the form $h = -\Delta + U$ on $\mathbf{R}^d$ with periodic potential U. Recently, Spohn[6] studied their long time behavior. To obtain the corresponding Euler equations in this case is clearly of interest but some problems immediately arise.

The first one is related to the identification and the proper interpretation of the corresponding Wigner distribution function. In other words, by what to replace Eqs. (2.7)–(2.9)? We do not know a unique good answer to that question (but see the discussion in refs. 11–13 and 20. An obvious generalization of (2.4) goes as follows.

Let ψ_{nk} be the eigenfunctions of the Hamiltonian $H = -\Delta + U$, with periodic potential U, having eigenvalues $E_n(k)$. Introduce the generalized Fourier transformation (cf. ref. 21)

$$\tilde{\chi}(k, n) = \int_{\mathbf{R}^d} dx\, \psi_{nk}(x)\, \chi(x) \tag{7.1}$$

We consider now an ensemble (we denote creation and annihilation operators by $a^*(x)$, $a(y)$) of particles subject to a periodic potential U and a family of states ω^ε for which the following limit exists (Λ^* is the first Brillouin zone)

$$f_n(r, k) = \lim_{\varepsilon \downarrow 0} \varepsilon \int_{\mathbf{R}^d} dv\, e^{-irv} \omega^\varepsilon \times \left[\tilde{a}^*\left(k + \varepsilon\frac{v}{2}, n\right) \tilde{a}\left(k - \varepsilon\frac{v}{2}, n\right) \right], \qquad r \in \mathbf{R}^d, \quad k \in \Lambda^* \tag{7.2}$$

and call it the nth band distribution function of the ensemble. $\tilde{a}(k, n)$ is to be understood as (7.1), but now as operators. We recover (2.7) at time $t = 0$ if in (7.1)–(7.2) we use $\psi_{nk}(x) = (1/2\pi)\, e^{-ikx}$. Comparing with (2.8)–(2.9), we have

$$\int dk\, f_n(r, k) = \lim_{\varepsilon \downarrow 0} \omega^\varepsilon(a^*(\varepsilon^{-1} r, n)\, a(\varepsilon^{-1} r, n)) \tag{7.3}$$

and

$$\int dr\, f_n(r, k) = \lim_{\varepsilon \downarrow 0} \varepsilon \omega^\varepsilon(\tilde{a}^*(k, n)\, \tilde{a}(k, n)) \tag{7.4}$$

Proposition 4. Let $\alpha_t(\cdot)$ be the time evolution of the ensemble generated by the one particle Hamilton operator $H = -\Delta + U$ on $\mathbf{R}^d$: $H\psi_{nk} = E_n(k)\, \psi_{nk}$. Then,

$$f_n(r, k; t) = \lim_{\varepsilon \downarrow 0} \varepsilon \int_{\mathbf{R}^d} dv\, e^{-irv} (\omega^\varepsilon \circ \alpha_{\varepsilon^{-1} t}) \times \left[\tilde{a}^*\left(k + \varepsilon\frac{v}{2}, n\right) \tilde{a}\left(k - \varepsilon\frac{v}{2}, n\right) \right] \qquad r \in \mathbf{R}^d, \quad k \in \Lambda^* \tag{7.5}$$

exists and

$$\partial_t f_n(r, k; t) + \partial_k E_n(k) \cdot \partial_r f_n(r, k; t) = 0 \tag{7.6}$$

Proof. This is an easy task since $\alpha_t \tilde{a}(k, n) = e^{-itE_n(k)} \tilde{a}(k, n)$ and $f_n(r, k; t) = f_n(r - \partial_k E_n(k)\, t, k)$. ∎

ACKNOWLEDGMENTS

We have very much benefited from discussions with B. Baumgartner, M. Fannes, M. Moser, O. Penrose, especially with H. Spohn, Yu. Suhov, and A. Verbeure. We very much thank an anonymous referee for important suggestions and remarks. W.S. was supported by the project P10517-NAW of the Österreichische Forschungsfond, FWF. He also wants to thank the Institute of Theoretical Physics in Leuven for their very warm hospitality and financial support.

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